ALLOCATION OF TWO TYPES OF AIRCRAFT
IN TACTICAL AIR WAR:

A GAME-THEORETIC ANALYSIS

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SUMMARY

The problem of allocating two types of aircraft (bombers and fighters) among three different air tasks (counter air, air defense, and support of ground operations) in a multi-strike campaign is analyzed as a two-sided war game. It is assumed that a bomber can be used in either the counter air or ground support operations, while a fighter can be used in either the air defense or ground support roles. That is, bombers and fighters have one task—ground support—in common.

Optimal employment during the last strikes of the campaign consists of a concentration of all resources on support of ground operations. Optimal employment during the early strikes of the campaign requires randomization by both sides.

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ALLOCATION OF TWO TYPES OF AIRCRAFT IN TACTICAL AIR WAR: A GAME-THEORETIC ANALYSIS

1. INTRODUCTION

An important allocation problem associated with a tactical air campaign is that of allocating the tactical air forces among the various air tasks in a competitive environment. Taking into account the enemy's possible allocations, we wish to determine an optimal employment of tactical resources. Stated in this form, this allocation problem is a problem in the theory of games of strategy.

In a previous paper [1], we studied the employment of tactical air forces in the various theater air tasks by formulating the problem as a zero-sum two-person game. It was assumed there that both sides had one type of aircraft to be allocated among three tasks; counter air, air defense, and close support.

The game formulated in the present paper assumes that each side has two different types of aircraft to which we give the generic names of bomber and fighter. A bomber can only be used in either the counter air or ground support roles, and a fighter only in the air defense or ground support roles.

The introduction of two types of aircraft into the tactical game yields substantially different and more complex optimal tactics even for campaigns of short duration. In [1] it is shown that if the allocations are restricted to counter air and close support tasks, then both sides have optimal pure strategies; if a third task is added, then one side has an optimal pure strategy and the other a mixed strategy. In contrast, if two types of aircraft are to be allocated among three tasks, then both sides may have optimal mixed strategies. As an example of the complexity of the last case, we shall present the optimal tactics, i.e., the optimal employment of aircraft, for a relatively short campaign.

2. FORMULATION OF TACTICAL WAR GAME

The tactical air war game is viewed as a series of strikes, or moves, each of which consists of simultaneous counter air, air defense and ground support operations. These operations are undertaken by each side in order to accomplish a given theater mission or payoff.

Suppose that at the start of the campaign one side, which we call Blue, has an air force consisting of B bombers and F fighters, while his opponent, Red has β bombers and Φ fighters. Each bomber may be used on counter air or ground support missions. Each fighter may be used on air defense or ground support missions.

Let us now examine a strike in the campaign, say the initial strike. Suppose Blue dispatches x of his bombers on counter air operations, and the remainder, $B - x$, on ground support operations. Some of these x bombers, say rx of them, where $0 \leq r \leq 1$, are dispatched against enemy bomber fields and the remainder, $(1 - r)x$, against enemy fighter fields. Suppose that during this strike Blue also sends u of the F

fighters on air defense missions and the remainder, $F - u$ fighters, on ground support operations. Hence $B + F - x - u$ Blue planes participate in ground support operations during this initial strike.

Similarly, suppose that on the initial strike Red allocates ξ bombers to counter air operations, and $\beta - \xi$ bombers to ground support operations. Of these ξ bombers, let $\rho\xi$ of them, where $0 \leq \rho \leq 1$, be assigned to attack enemy bomber fields and the remainder, $(1 - \rho)\xi$ bombers, to attack enemy fighter fields. Suppose that Red also assigns μ of the Φ fighters to air defense missions and the remainder, $\Phi - \mu$ fighters, to ground support missions. The number of Red planes participating in ground support operations during this initial strike is therefore $\beta + \Phi - \xi - \mu$.

In making their allocations the players know the size of their own and opponent's forces. However, neither side knows the allocation made by his opponent until after the strike is completed.

Let us now describe the outcome of the above strike. The fighters that Red allocates to air defense will reduce the number of Blue bombers that penetrate to counter-air targets, but will not affect ground support operations. The number of attacking planes that are prevented from reaching their targets will be assumed to be proportional to μ, say $c\mu$, unless Blue's attacking planes are saturated, i.e., if $c\mu$ exceeds x. The constant c is called the air defense potential, as it measures the effectiveness of the air defense aircraft. We assume that the Red fighters are unable to distinguish which of the Blue attacking bombers are destined for Red bomber bases and which are destined for Red fighter bases. The interceptions are assumed to be distributed uniformly at random among the attacking aircraft. Thus, the number of Blue bombers penetrating Red's defense is $x - c\mu$ as long as $c\mu$ is not larger than x. If $c\mu$ is larger than x, no Blue bombers penetrate. Thus the number of Blue bombers penetrating Red's defenses is given by

$$\max\,(0,\ x - c\mu).$$

Of the penetrating bombers, $r \max\,(0,\ x - c\mu)$ will attack Red bomber fields and $(1 - r) \max\,(0,\ x - c\mu)$ will attack Red fighter fields. Therefore the number of Blue bombers attacking Red bomber fields is

$$\max[0,\ r(x - c\mu)], \qquad 0 \le r \le 1,$$

and the number of Blue bombers attacking Red fighter fields is

$$\max\,[0,\ (1 - r)(x - c\mu)] \qquad 0 \le r \le 1.$$

The Blue bombers that penetrate to target destroy parked enemy aircraft by dropping bombs on the Red airfields. Let us assume that each of Blue's penetrating bombers can destroy b_1 enemy bombers or b_2 enemy fighters, and let us further assume that all of Red's aircraft are at risk at the time of a strike. Then the number of Red bombers destroyed by Blue on the initial strike is:

$$\min\left\{\beta,\ b_1 \max\,[0,\ r(x - c\mu)]\right\},$$

and the number of Red fighters destroyed is

$$\min \left\{\Phi,\ b_2 \max\ [0,\ (1 - r)(x - c\mu)]\right\} .$$

We shall assume that the losses the Red air force suffers from accidents and ground defenses are small, and will be neglected in our analysis. We further assume that planes used in air defense and ground support suffer no losses, and that Red bombers failing to penetrate the Blue air defense return to base. That is to say, we assume that losses suffered in the air battle are negligible compared to the other losses, and that air defense aircraft prevent attacking planes from successfully delivering their bombs without necessarily destroying the bombers. Thus, we see that during the initial strike Red's bomber force is reduced to

$$\beta_1 = \max \left\{0,\ \beta - b_1 \max\ [0,\ r(x - c\mu)]\right\} ,$$

and Red's fighter force is reduced to

$$\Phi_1 = \max \left\{0,\ \Phi - b_2 \max\ [0,\ (1 - r)(x - c\mu)]\right\} .$$

In exactly the same manner we can analyze the effect of this initial strike on Blue's aircraft inventories. At the end of the initial strike, the number of bombers available to Blue is

$$B_1 = \max \left\{0,\ B - d_1 \max\ [0,\ \rho(\xi - eu)]\right\} ,$$

and the number of fighters is

$$F_1 = \max \left\{0,\ F - d_2 \max\ [0,\ (1 - \rho)(\xi - eu)]\right\} ,$$

where e, d_1 and d_2 have similar meanings as c, b_1 and b_2, respectively.

Blue now has B_1 bombers and F_1 fighters and Red has β_1 bombers and Φ_1 fighters that they can allocate for the second strike. This strike will result in new inventories B_2, F_2, β_2, Φ_2 for the third strike. This process is repeated for the duration of the campaign, which consists of a predetermined number of strikes.

3. PAYOFF

In our model the function of the tactical air force is to assist the ground forces, and the results will vary with the number of planes allocated to ground support operations. On the other hand, the enemy also can aid his ground forces by allocating aircraft to ground support operations. Thus some of the assistance might "cancel out." We shall assume that the assistance offered to Blue ground forces, or payoff to Blue, on a given strike can be measured by the difference between Blue ground support sorties and Red ground support sorties, namely

$$(B + F - x - u) - (\beta + \Phi - \xi - \mu).$$

Implicit here is the assumption that bombers and fighters are equally effective in the ground support role. The payoff, M, to Blue for the entire campaign of N strikes is the sum of these scores for each of the N strikes, or

$$M = \sum^{N} [(B + F - x - u) - (\beta + \Phi - \xi - u)].$$

The problem faced by each side is now apparent. For example, Blue would like to allocate a large number of planes to ground support missions and thereby increase the payoff at a given move, yet he would like to destroy the Red air force by means of counter air operations in order to ensure that Red will not be able to mount any ground support sorties in subsequent strikes. Further, if Blue does not provide for air defense he may suffer severe losses to his own air force if Red elects to mount a large counter air strike. Each player has to take the future as well as the possibilities open to his opponent into account.

4. PARTICULAR PARAMETER VALUES

From the description of the game and the definition of the payoff function, it is clear that the optimal tactics will depend on the values of the constants b_1, b_2, c, e, d_1, d_2 as well as on the magnitude of the initial inventories, which are B and F for Blue and β and Φ for Red. Since our main purpose in this paper is to illustrate how the optimal allocations of two types of aircraft vary with the size of the initial inventories, we shall assign particular values to the six constants. For computational convenience we shall let $b_1 = b_2 = c = e = d_1 = d_2 = 1$.

The inventory of Blue planes at the end of a strike now becomes

$$B_1 = \max\left\{0,\ B - \max\left[0,\ \rho(\xi - u)\right]\right\}$$

$$F_1 = \max\left\{0,\ F - \max\left[0,\ (1-\rho)(\xi - u)\right]\right\}\ ,$$

and the inventory of Red planes becomes

$$\beta_1 = \max\left\{0,\ \beta - \max\left[0,\ r(x - \mu)\right]\right\}$$

$$\Phi = \max\left\{0,\ \Phi - \max\left[0,\ (1-r)(x - \mu)\right]\right\}\ .$$

Finally we assume that at the start of the campaign, the fighter strength of the two air forces are equal, that is $F = \Phi$. To facilitate the analysis of the problem let us take $F = \Phi = 1$ at the start of the campaign; this merely means that we are changing our unit of measurement from the individual airplane to the total fighter force at the start of the campaign.

Although we have formulated the game for an arbitrary number of strikes, we shall present the optimal tactics for campaigns of three strikes. The complete mathematical proofs of these results are given in [2]. However, the technique used to derive the results is sketched in the Appendix, together with an outline of the method of proof. A precise mathematical formulation of the game in normal form is also given in the Appendix.

5. OPTIMAL TACTICS — EQUAL STRENGTH

We begin by describing the optimal tactics for the case in which Red and Blue initially have the same strengths, that is, when $B = \beta$ and $F = \Phi = 1$. Further, we shall restrict ourselves to a campaign of three strikes.

First consider the initial move of the game. The nature of the optimal tactic depends on the bomber strength. Let us denote the value of the common bomber strength by k; i.e., $B = \beta = k$. Then the optimal tactics at the first move are as follows.

Optimal Tactics at Initial Strike

If $k \geq 2$, then each side uses all of its bombers on counter air and all of its fighters on air defense. Of the bombers assigned to counter air targets, the fraction assigned to enemy bomber fields is $k/(k + 1)$, with $1 - k/(k + 1)$ of the bombers assigned to enemy fighter fields.

If $1 \leq k \leq 2$, then each side must bluff, i.e., randomize, over three tactics. Each side sends all bombers on counter air and all fighters on air defense with probability $(k - 1)/2$. Each side sends all bombers on counter air and all fighters on ground support with probability $(2 - k)/2$. Each side sends all bombers on ground support and all fighters on air defense with probability 1/2. Of the bombers assigned to counter air targets, the fraction assigned to bomber fields is $k/(k + 1)$, with the rest going against fighter fields.

If $k \leq 1$ each side must again bluff, but now only over two choices. This time, the tactic of all bombers on counter air and all fighters on air defense is chosen with probability one-half and the tactic of all bombers and all fighters on ground support is chosen also with probability one-half. Again, of the bombers allocated to counter air, the fraction assigned to enemy bomber fields is $k/(k + 1)$, with the rest attacking fighter fields.

Optimal Tactics for Last Two Moves

The optimal tactics for the last two moves of the game are the same for each player, and consist of concentrating all resources on support of ground operations.

The interesting feature of the optimal tactics, is that although the strengths of the two sides are the same, both players must randomize, or bluff, for an appreciable range of initial conditions. This is quite different from the optimal tactics in the corresponding situation when we have only one type of aircraft capable of performing any type of mission. In the latter case, when the two sides are of equal strength, each side has a pure strategy, and there is no need to bluff.

6. OPTIMAL TACTICS - UNEQUAL BOMBER STRENGTHS

We shall now drop the assumption that Red and Blue have equal bomber forces at the start of the campaign. The assumption that $F = \Phi = 1$ still holds. Since the effectiveness of the Red and Blue aircraft are the same, to describe the optimal tactics, we can restrict ourselves to the situation in which the Blue bomber force is initially larger than or equal to the Red bomber force. That is $B \geq \beta$. If the converse is true, namely, $\beta \geq B$, the optimal tactics are obtained from the case $B \geq \beta$ by interchanging the roles of Blue and Red.

In order to describe the optimal tactics, it is necessary to consider several cases, according to the difference between

Blue and Red bomber strengths at the start of the campaign. We therefore define

$$m = B - \beta$$

The nature of the optimal tactic will depend on the relative strengths of Red and Blue. Each possible combination of initial Blue and Red bomber strengths is given by a pair of numbers (B, β), and hence can be represented by a point in the (B, β) plane with $B \geq \beta$. Figure 1 presents a decomposition of the (B, β) plane into nine regions. We shall describe the optimal tactics at the initial move for each region.

Optimal Tactics at Initial Move

The optimal allocations of aircraft to the various tasks at the initial move is given in Table 1. Of the Blue bombers sent against counter air targets, the fraction sent against bomber fields is $\beta/(\beta + 1)$, and the fraction sent against fighter fields is $1/(\beta + 1)$. Of the Red bombers sent against counter air targets, the fraction assigned to bomber fields is $B/(B + 1)$ and the fraction assigned to fighter fields is $1/(B + 1)$.

Optimal Tactics During Last Two Moves

The optimal allocation for each side during the last two moves is to assign all aircraft to ground support operations.

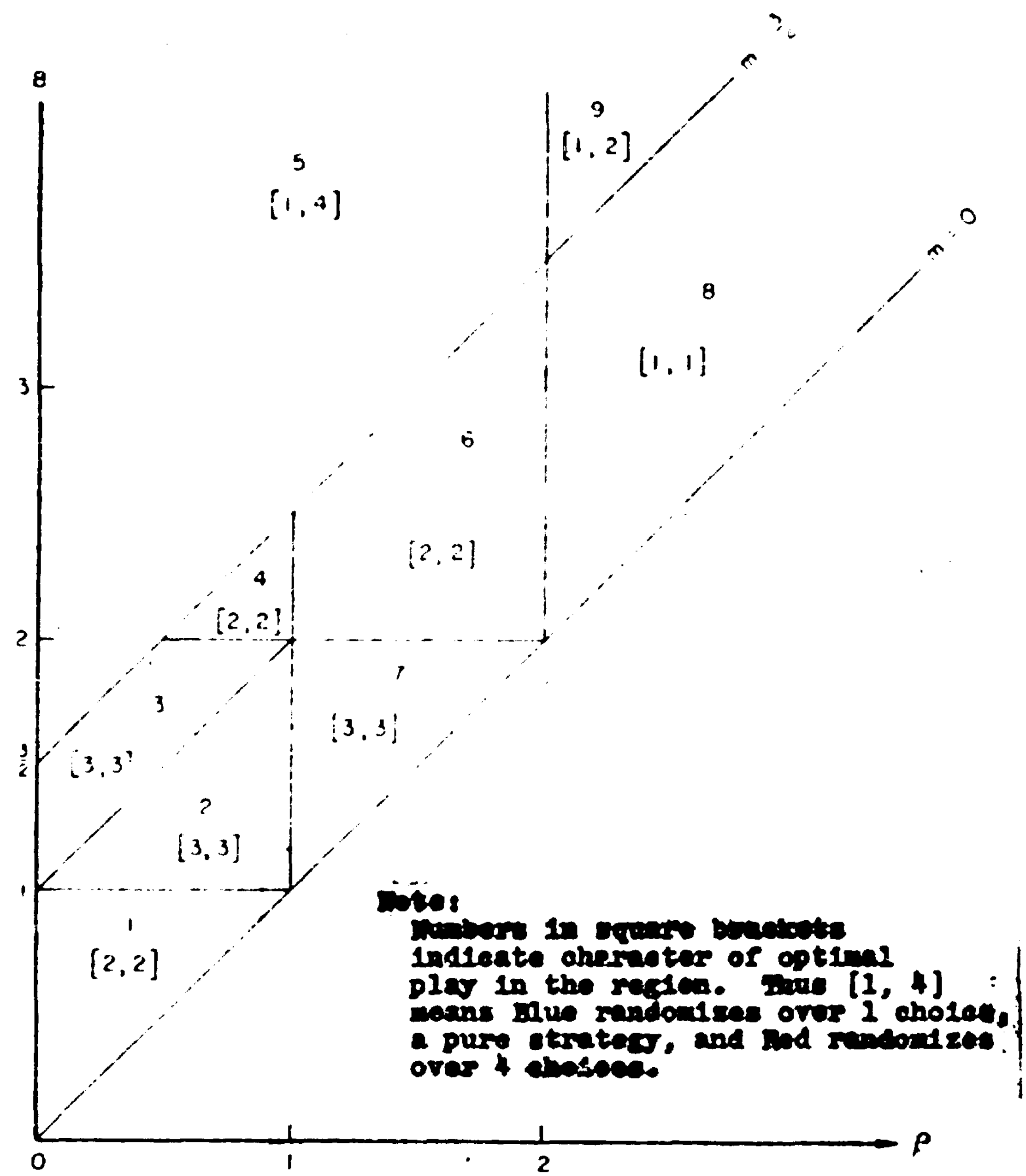

Fig 1 — Regions of (B, β) plane

Table 1

VALUE OF GAME AND OPTIMAL CHOICES FOR INITIAL MOVE

Region of (B, β) Plane (See Fig. 1)		Value of Game	Blue Optimal Choice		Red Optimal Choice	
No.	Description		Choice (x, u)	Prob.	Choice (ξ, μ)	Prob.
1	$0 \le m \le 1$	$\frac{7}{2} m$	(B, β)	$\frac{1}{2}$	(β, B)	$\frac{1}{2}$
	$0 \le \beta \le 1$		(0, 0)	$\frac{1}{2}$	(0, 0)	$\frac{1}{2}$
	$0 \le B \le 1$					
2	$0 \le m \le 1$	$\frac{7}{2} m$	(B, β)	$\frac{1}{2}$	(β, 1)	$\frac{B-1}{2}$
	$0 \le \beta \le 1$		(B, 0)	0	(β, 0)	$\frac{2-B}{2}$
	$1 \le B \le 2$		(0, 0)	$\frac{1}{2}$	(0, 1)	$\frac{1}{2}$
3	$1 \le m \le \frac{3}{2}$	$\frac{7}{2} m + \frac{(m-1)(B-2)}{2(2-m)}$	(B, β)	$\frac{1}{2}$	(β, 1)	$\frac{\beta}{2(2-m)}$
	$0 \le \beta \le 1$		(B, 0)	$\frac{m-1}{2(2-m)}$	(β, 0)	$\frac{2-B}{2(2-m)}$
	$1 \le B \le 2$		(0, 0)	$\frac{3-2m}{2(2-m)}$	(0, 1)	$\frac{1}{2}$
4	$1 \le m \le \frac{3}{2}$	$4m + \frac{\beta}{2} - 1$	(B, β)	$\frac{1}{2}$	(β, 1)	$\frac{1}{2}$
	$\frac{1}{2} \le \beta \le 1$		(B, 0)	$\frac{1}{2}$	(0, 1)	$\frac{1}{2}$
	$2 \le B \le \frac{5}{2}$					
5	$\frac{3}{2} \le m$	$3m + \frac{\beta}{2} + \frac{1}{2}$	$\left(\beta + \frac{3}{2}, \frac{\beta}{2}\right)$	1	(β, 1)	$\frac{1}{4}$
	$0 \le \beta \le 2$				(β, 0)	$\frac{1}{4}$
					(0, 1)	$\frac{1}{4}$
					(0, 0)	$\frac{1}{4}$
6	$0 \le m \le \frac{3}{2}$	$4m + \frac{\beta}{2} - 1$	(B, 1)	$\frac{\beta}{2}$	(β, 1)	$\frac{1}{2}$
	$1 \le \beta \le 2$		(B, 0)	$1 - \frac{\beta}{2}$	(0, 1)	$\frac{1}{2}$
	$2 \le B \le \frac{7}{2}$					

Table 1—Continued

Region of (B, β) Plane (See Fig. 1)		Value of Game	Blue Optimal Choice		Red Optimal Choice	
No.	Description		Choice (x, u)	Prob.	Choice (ξ, μ)	Prob.
7	$0 \le m \le 1$ $1 \le \beta \le 2$ $1 \le B \le 2$	$\frac{7}{2} m$	$(B, 1)$ $(B, 0)$ $(0, 1)$	$\frac{\beta - 1}{2}$ $\frac{2 - \beta}{2}$ $\frac{1}{2}$	$(\beta, 1)$ $(\beta, 0)$ $(0, 1)$	$\frac{B - 1}{2}$ $\frac{2 - B}{2}$ $\frac{1}{2}$
8	$0 \le m \le \frac{3}{2}$ $2 \le \beta$	$4m$	$(B, 1)$	1	$(\beta, 1)$	1
9	$\frac{3}{2} \le m$ $2 \le \beta$	$3m + \frac{3}{2}$	$\left(\beta + \frac{3}{2}, 1\right)$	1	$(\beta, 1)$ $(\beta, 0)$	$\frac{1}{2}$ $\frac{1}{2}$

7. DISCUSSION OF OPTIMAL TACTICS

That portion of the (B, β) plane for which $0 \leq m \leq 3/2$ can be thought of as being the set of initial conditions for which the opponents are about equal in strength. For, in the part of the plane where $0 \leq m \leq 3/2$, one can say that "both players do the same thing," if by doing the "same thing" is meant randomizing over the same number of choices. In some of the regions in the strip $0 \leq m \leq 3/2$, namely Regions 1, 7, and 8, the similarity of the strategies of the two players is even more marked. These regions have segments of the line $m = 0$ as part of their boundary and the character of the optimal strategies on these segments is preserved throughout the regions. In fact, in these regions, both players have the same strategies, just as they do on the line $m = 0$. In Region 8, each player uses all his bombers to attack counter air targets and all his fighters on air defense. In Region 7, each player randomizes over the following three tactics: all bombers to attack and all fighters on defense, all bombers to attack and all fighters on ground support, all bombers to ground support and all fighters to air defense. In Region 1 each player either goes "all out" on offense and defense, $x = B$, $u = \beta$ for Blue, and $\xi = \beta$, $\mu = B$ for Red, or goes "all out" for ground support operations by choosing $x = u = \xi = \mu = 0$, each tactic being chosen with probability 1/2. Since in Region 1 the bomber strength for each player is less than the opponent's fighter strength, there is no need to use all of the fighter force on air defense. Therefore, only enough fighters needed to match the largest possible number of incoming bombers are allocated to air defense. It should also be pointed out that Region 8 is the only region where both players have pure optimal tactics.

That portion of the (B, β) plane lying above the line $m = 3/2$, can be thought of constituting the set of initial conditions for which Blue is stronger than Red. If $m \geq 3/2$, Blue has a pure optimal tactic and Red must randomize.

Finally, we call attention to a curious phenomenon in Region 3. It is the only region in which the game value is not a linear function of B and β.

Appendix

1. MATHEMATICAL FORMULATION OF GAME

Let the moves be numbered from the end of the game; i.e., the n-th move means n moves to the end of the game. The players' allocations on the n-th move determine the state variables for the (n-1)-st move as follows:

$$
(1)\qquad
\begin{aligned}
B_{n-1} &= \max\left[0,\ B_n - \rho_n \max\left(0,\ \xi_n - u_n\right)\right] \\
F_{n-1} &= \max\left[0,\ F_n - (1-\rho_n) \max\left(0,\ \xi_n - u_n\right)\right] \\
\beta_{n-1} &= \max\left[0,\ \beta_n - r_n \max\left(0,\ x_n - \mu_n\right)\right] \\
\Phi_{n-1} &= \max\left[0,\ \Phi_n - (1-r_n) \max\left(0,\ x_n - \mu_n\right)\right].
\end{aligned}
$$

For the N-move game, the payoff to Blue is given by

$$
(2)\qquad \sum_{n=1}^{N} \left[(B_n + F_n - x_n - u_n) - (\beta_n + \Phi_n - \xi_n - \mu_n)\right].
$$

It is assumed that each player knows the manner in which the game proceeds; namely each player has the information expressed by equations (1). It is further assumed that at each stage of the game both players know the state variables and the entire past history of the play. That is, at the n-th move both players know N, B_N, F_N, β_N, Φ_N and x_i, u_i, r_i, ξ_i μ_i, ρ_i for $i = N, n-1, \ldots, n+2, n+1$. Since they also know equations (1) it follows that at the n-th move they know B_i, F_i, β_i, Φ_i for $i = N, N-1, \ldots, n+1, n$.

The pure strategies of the game in normal form will now be defined inductively on the number of moves in the game. First, a strategy for Blue in a one-move game is a choice of a point $X_1 = (x_1, u_1, r_1)$ in the cube $0 \leq x_1 \leq B_1$, $0 \leq u_1 \leq F_1$, $0 \leq r_1 \leq 1$. Similarly a strategy for Red is a choice of a point $Y_1 = (\xi_1, \mu_1, \rho_1)$ in the cube $0 \leq \xi_1 \leq \beta_1$, $0 \leq \mu_1 \leq \Phi_1$, $0 \leq \rho_1 \leq 1$. Let σ_N be a strategy for Blue in an N-move game; σ_N, of course, is a function of B_N, F_N, β_N, Φ_N. In an (N+1)-move game, at the (N+1)-st move Blue chooses a point $X_{N+1} = (x_{N+1}, u_{N+1}, r_{N+1})$ in the cube D_{N+1} defined by

$$0 \leq x_{N+1} \leq B_{N+1}, \quad 0 \leq u_{N+1} \leq F_{N+1}, \quad 0 \leq r_{N+1} \leq 1,$$

and simultaneously Red chooses a point $Y_{N+1} = (\xi_{N+1}, \mu_{N+1}, \rho_{N+1})$ in the cube Δ_{N+1} defined by

$$0 \leq \xi_{N+1} \leq \beta_{N+1}, \quad 0 \leq \mu_{N+1} \leq \Phi_{N+1}, \quad 0 \leq \rho_{N+1} \leq 1.$$

These choices yield state variables B_N, F_N, β_N, Φ_N by equations (1). A strategy σ_{N+1} for Blue in the N+1-move game is then defined as a choice X_{N+1} in D_{N+1} and a function Λ_N that associates to each point (X_{N+1}, Y_{N+1}) in the product space $D_{N+1}\,\Delta_{N+1}$ a strategy σ_N in the N-move game. Thus σ_{N+1} can be written as

$$\sigma_{N+1} = (X_{N+1}, \Lambda_N) .$$

Similarly a strategy τ_{N+1} for Red in the (N+1)-move game is defined as a choice Y_{N+1} and a function ψ_N that associates with each (X_{N+1}, Y_{N+1}) a strategy τ_N in the N-move game.

Thus τ_{N+1} can be written as

$$\tau_{N+1} = (Y_{N+1}, \psi_N) .$$

Mixed strategies for the players can now be defined in a similar manner. For a game of one move a mixed strategy for Blue is a probability distribution G_1 over the cube D_1, and a mixed strategy for Red is a probability distribution H_1 over the cube Δ_1. Suppose now that mixed strategies for games of length N have been defined. Let G_N be a mixed strategy for Blue in an N-move game. A mixed strategy G_{N+1} in a game of N+1 moves is a probability distribution g_{N+1} over the cube D_{N+1} and a function λ_N that associates to each (X_{N+1}, Y_{N+1}) a mixed strategy G_N in the N-move game. Thus the mixed strategy in the (N+1)-move game can be written as

$$G_{N+1} = (g_{N+1}, \lambda_N).$$

Mixed strategies H_{N+1} for Red are defined similarly by a distribution h_{N+1} or Δ_{N+1} and a function ψ_N, and can be written as

$$H_{N+1} = (h_{N+1}, \psi_N).$$

2. SUFFICIENT CONDITIONS FOR OPTIMAL STRATEGIES

Suppose that in the game of length N there exist strategies G_N^* for Blue and H_N^* for Red with the following properties:

(1) If Blue plays G_N^* and Red plays H_N^*, the expectation $E(G_N^*, H_N^*)$ exists.

(ii) For all Red pure strategies τ_N, $E(G_N^*, \tau_N)$ exists, and

$$E(G_N^*, \tau_N) \geq (G_N^*, H_N^*) \ .$$

(iii) For all Blue pure strategies σ_N, $E(\sigma_N, H_N^*)$ exists, and

$$E(\sigma_N, H_N^*) \leq E(G_N^*, H_N^*) \ .$$

In this event the game is said to have a value $V_N = V_N(B_N, F_N, \beta_N, \Phi_N)$ given by

$$V_N(B_N, F_N, \beta_N, \Phi_N) = E(G_N^*, H_N^*);$$

G_N^* is said to be an optimal strategy for Blue and H_N^* is said to be an optimal strategy for Red. The value, as indicated by the notation, is a function of the initial conditions B_N, F_N, β_N, Φ_N.

Define

$$L_{N+1}(X_{N+1}, Y_{N+1}) = B_{N+1} + F_{N+1} - x_{N+1} - u_{N+1} - \beta_{N+1} - \Phi_{N+1} + \xi_{N+1} + \mu_{N+1}$$

and

$$\;_{N+1}(X_{N+1}, Y_{N+1}) = L_{N+1}(X_{N+1}, Y_{N+1}) + V_N(B_N, F_N, \beta_N, \Phi_N),$$

where B_N, F_N, β_N, Φ_N are obtained from B_{N+1}, F_{N+1}, β_{N+1}, Φ_{N+1} by means of (1) and the choices (X_{N+1}, Y_{N+1}). We can now state a Lemma that enables us to solve the game inductively.

<u>Lemma 1</u>. Let the game of length N have value $V_N(B_N, F_N, \beta_N, \Phi_N)$, with optimal strategies G_N^* and H_N^* for Blue

and Red, respectively. Let g^*_{N+1} be a distribution in D_{N+1} and h^*_{N+1} a distribution on Δ_{N+1} such that

$$(3)\quad \int M_{N+1}(X_{N+1}, Y_{N+1})dg^*_{N+1} \geq \iint M_{N+1}(X_{N+1}, Y_{N+1})dg^*_{N+1}dh^*_{N+1} \quad \text{for all } Y_{N+1},$$

$$(4)\quad \int M_{N+1}(X_{N+1}, Y_{N+1})dh^*_{N+1} \leq \iint M_{N+1}(X_{N+1}, Y_{N+1})dg^*_{N+1}dh^*_{N+1} \quad \text{for all } X_{N+1}.$$

Then the game of length N+1 has value

$$V_{N+1}(B_{N+1}, F_{N+1}, \beta_{N+1}, \Phi_{N+1}) = \iint M_{N+1}(X_{N+1}, Y_{N+1})dg^*_{N+1}dh^*_{N+1},$$

and the optimal strategies are

$$G^*_{N+1} = (g^*_{N+1}, G^*_N) \text{ for Blue}, \quad H^*_{N+1} = (h^*_{N+1}, H^*_N) \text{ for Red}.$$

The proof of this lemma is the same as that given for the analogous result in [1] and therefore will not be repeated here.

3. SOLUTION FOR N = 1, 2

For $N = 1$, an examination of the payoff (2) shows that both Blue and Red have optimal pure strategies consisting of the choices $x_1 = u_1 = 0$ for Blue, and $\xi_1 = \mu_1 = 0$ for Red. The choices of r_1 and ρ_1 are clearly arbitrary in this case. The value of the game in this instance is $V_1 = B_1 + F_1 - \beta_1 - \Phi_1$.

It follows from Lemma 1 that for $N = 2$ it suffices to consider

$$M_2(X_2, Y_2) = B_2 + F_2 - \beta_2 - \Phi_2 - x_2 - u_2 + \xi_2 + \mu_2 + [B_1 + F_1 - \beta_1 - \Phi_1] .$$

By means of equations (1), the quantity in square brackets can be expressed in terms of quantities involving the subscript 2. When the coefficients of the quantities x_2, u_2 ξ_2, μ_2 are then examined, it is seen that the optimal choice for Blue is $x_2 = u_2 = 0$, r_2 arbitrary, and the optimal choice for Red is $\xi_2 = \mu_2 = 0$, ρ_2 arbitrary. The value of the two-move game is $V_2 = 2(B_2 + F_2 - \beta_2 - \Phi_2)$.

4. SOLUTION FOR N = 3

It follows from Lemma 1 that for $N = 3$ it suffices to consider the game with payoff

$$M_3(X_3, Y_3) = B_3 + F_3 - \beta_3 - \Phi_3 - x_3 - u_3 + \xi_3 + \mu_3 + 2[B_2 + F_2 - \beta_2 - \Phi_2].$$

It can be shown that since the optimal choices for $i = 1, 2$ are $x_1 = u_1 = \xi_1 = \mu_1 = 0$, then the optimal choice of r_3 is $\beta_3/(\beta_3 + \Phi_3)$, and the optimal choice of ρ_3 is $B_3/(B_3 + F_3)$. The rest of the proof consists of verifying that the strategies given in Table 1, do indeed satisfy (3) and (4). We note here that since we first determine the optimal r_3 and ρ_3, we can now take $X_3 = (x_3, u_3)$, $Y_3 = (\xi_3, \mu_3)$ in (3) and (4). The verification process, although rather long and tedious, is fairly straightforward. It is presented in [2].

A more difficult problem than that of verifying, is that of guessing the optimal strategies, or stated in another way, that of selecting the candidates for verification. In the case of equal initial bomber strengths the following procedure was used.

The trial assumption was first made that the optimal strategies would consist of step functions with at most a finite number of jumps and that these jumps would occur at extreme points of the strategy spaces. A finite matrix game was then set up with payoff $M_3(X_3, Y_3)$ and strategies chosen from the extreme points of the strategy spaces D and Δ. The solution of the resulting matrix game was then determined, and the optimal strategies found for the matrix game were tested for optimality in the full game by substituting them in (3) and (4). If $k \leq 1$, however, it is clear from heuristic grounds that some of the extreme points are dominated by other boundary points. For example, from the structure of the game it is clear that if $k \leq 1$, then (k, 1) wastes some of Blue's fighters and (k, k) should dominate (k, 1). The matrix game was therefore expended in this case to include such strategies.

In the case of unequal initial bomber strengths the method used to guess the optimal strategies was a combination of the method used in the symmetric case, namely, setting up a finite matrix game, and what might be described as "contagion and continuity." That is, clues as to the nature of the optimal strategies were obtained from the knowledge of the solution in contiguous regions, and these were used to guess the optimal strategies or to modify the appropriate matrix games. This process was especially useful when one could determine the value of the game in a region by continuity from knowledge of the values in adjacent regions.

REFERENCES

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2. ———————————, A Multimove Allocation Game, The RAND Corporation, Paper P-1533, October, 1958.